# STACK OF COHERENT SHEAVES

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Fix a map $f\colon X \to S$ of schemes so that

(1) $S$ is noetherian.
(2) $f$ is projective.
(3) For each cartesian diagram of schemes

$$\begin{tikzcd} X_T \arrow[r] \arrow[d, "f_T"'] & X \arrow[d, "f"] \\ T \arrow[r] & S \end{tikzcd}$$

we have $(f_T)_*\mathcal{O}_{X_T} \cong \mathcal{O}_T$.

In the situation above we define a category fibered in groupoids (CFG) of coherent sheaves, defined $\mathrm{Coh}_{X/S}$ as follows. For each $T \in \mathrm{Sch}_S$ we assign the groupoid

$$\mathrm{Coh}_{X/S}(T) = \left\{ \text{Groupoid of coherent sheaves on } X_T \text{ which are flat over } T \right\}.$$

Our aim in this note is to explain the following theorem

**Theorem 0.1.** *With hypothesis as above, the stack $\mathrm{Coh}_{X/S}$ is an Artin stack, locally of finite type and with affine diagonal.*

*Remark* 0.2. The $T$-flatness condition of $\mathcal{F}_T$ along with properness of $f_T$, guarantee that $R(f_T)_*F$ is a perfect complex on $T$ and thus satisfies several local constancy and semi-continuity conditions on the base.

*Example* 0.3. Let $f\colon X \to \mathrm{Spec}(k)$ be a smooth projective and geometrically connected curve. Then the conditions on $f$ are easily seen to be satisfied. Then for each $k$-scheme $U$, we see that $\mathrm{Coh}_{X/k}(U)$ can be interpreted as a category of coherent sheaves on the relative curve $X \times_k U$ (since the base change of a smooth morphism is smooth, hence flat).

*Example* 0.4. Fix a positive integer $n$ and consider the substack $\mathrm{Fib}_{X/S,n}$ of $\mathrm{Coh}_{X/S}$ which now assigns to each $S$-scheme $U$, the groupoid of locally free sheaves of rank $n$ on $X \times_S U$. Since rank is upper semi-continuous and local freeness is open, we see that this is an open substack of $\mathrm{Coh}_{X/S}$ (once algebraicity of the latter has been shown) of finite type over $S$. In fact without fixing rank it is easy to see that $\mathrm{Fib}_{X/S}$ is an open substack locally of finite type.

*Example* 0.5. When $f\colon X \to \mathrm{Spec}\, k$ is as in the previous example, then this stack is denoted $\mathrm{Bun}_{\mathrm{GL_n}}(X)$ and is a baby example of the sort of stacks one works with in the geometric Langlands program. This stack is a smooth Artin stack over $k$, a fact which is no longer true as soon as $X$ is higher dimensional, for example a surface. We invite the reader to check this by looking at the rank of the tangent space (suitably defined) at closed points.

Now by étale descent of coherent sheaves we observe that $\mathrm{Coh}_{X/S}$ is a stack over $(\mathrm{Sch}_S)_{et}$. We want to show that in fact it is an algebraic stack, of finite type over $S$.

**Definition 0.6.** A stack $\mathscr{X}/S$ is an algebraic stack if the following hold:

(1) The diagonal $\Delta\colon \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable, separated and quasi-compact.
(2) There exists a smooth surjective morphism $X \to \mathscr{X}$ with $X$ a scheme.

*Remark* 0.7. We remark that it is most natural to use Artin's criteria to prove the representability of this stack, but we prefer here to use the above definition in interest of getting some intuition for the geometry involved.

*Remark* 0.8. The definition as written above is different from the one in Alper's notes. The difference is that we do not require the presentation $X \to \mathscr{X}$ to be representable. However, the condition of representability of the diagonal immediately guarantees that any presentation (and in fact any morphism from a scheme) is representable. The converse that requiring representability of the presentation implying representability of the diagonal is also true but harder.

In practice, it is hard to verify that a given presentation is representable, while the condition on the diagonal can be verified with relatively little effort as the next lemma implies.

**Lemma 0.9.** *Let $\mathscr{X}/S$ be a stack. Then the diagonal is representable if and only for each $T \to S$, and $x, y \in \mathscr{X}(T)$ the sheaf* $\mathrm{Isom}_{\mathscr{X}}(x, y)$ *is an algebraic space.*

*Proof.* Consider the cospan

$$\begin{tikzcd} & T \arrow[d, "{(x,y)}"] \\ \mathscr{X} \arrow[r, "\Delta_{\mathscr{X}}"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd} .$$

Observe that the fibre product $\mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} T$ consists triples $((\alpha, \alpha), (f : W \to T), (\varphi_x, \varphi_y))$ where $\varphi_x : f^*x \to \alpha$ and $\varphi_y : f^*y \to \alpha$ are isomorphisms. This gives an isomorphism $\varphi_{yx} := \varphi_y^{-1} \circ \varphi_x : f^*x \to f^*y$ which has to come from a morphism $x \to y$. Similarly any morphism $\eta_{xy} \in \mathrm{Isom}(x, y)$ gives a triple in the fibre product. So we have a canonical identification $\mathrm{Isom}_{\mathscr{X}}(x, y)$ with $\mathscr{X} \times_{\mathscr{X} \times_S \mathscr{X}} T$. Then the result follows by definition of representability of morphisms. $\square$

We now verify that the diagonal $\Delta : \mathrm{Coh}_{X/S} \to \mathrm{Coh}_{X/S} \times_S \mathrm{Coh}_{X/S}$ is representable. It suffices to show that for each scheme $T \to S$ and sheaves $\mathcal{F}$ and $\mathcal{G}$ on $X_T$ flat over $\mathcal{O}_T$, the functor $\mathrm{Sch}_T \to \mathrm{Sets}$ sending $V$ to $\mathrm{Isom}_{\mathcal{O}_V}(\mathcal{F}_V, \mathcal{G}_V)$ is representable in algebraic spaces. In fact, we will show that this functor is representable in schemes.

**Lemma 0.10.** *The functor* $\mathrm{Isom}(\mathcal{F}, \mathcal{G}) : \mathrm{Sch}_T \to \mathrm{Sets}$ *described above is relatively representable in schemes by a relatively affine scheme over $S$.*

*Proof.* We explain a quick proof which moreover highlights the style of argument one often uses in proving that the $\mathrm{Isom}$ sheaf is representable. By noetherian approximation (which we leave the reader to explicate) we may reduce to the case where $T$ is finite type over $S$ and hence noetherian and $\mathcal{F}$ and $\mathcal{G}$ are noetherian. In this case $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ are representable in $\mathrm{Sch}_T$ by a relatively affine $T$ scheme $\mathbb{V}(\mathcal{E})$ for some coherent sheaf $\mathcal{E}$ on $T$. Note that a similar result holds for $\mathcal{H}om(\mathcal{F}, \mathcal{F})$, $\mathcal{H}om(\mathcal{G}, \mathcal{G})$ and $\mathcal{H}om(\mathcal{G}, \mathcal{F})$. We claim that the following diagram where $\mathrm{comp}(\alpha, \beta) = (\alpha \circ \beta, \beta \circ \alpha)$ is cartesian

$$\begin{tikzcd} \mathrm{Isom}(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] & \mathcal{H}om(\mathcal{F}, \mathcal{G}) \times_T \mathcal{H}om(\mathcal{G}, \mathcal{F}) \arrow[d, "\mathrm{comp}"] \\ T \arrow[r, "{(id,id)}"] & \mathcal{H}om(\mathcal{F}, \mathcal{F}) \times_T \mathcal{H}om(\mathcal{G}, \mathcal{G}) \end{tikzcd} ,$$

and since every vertex is representable, so is the pullback. In fact more can be said, since the morphism in the bottom arrow is a section of a separated morphism, it is a closed immersion and hence $\mathrm{Isom}(\mathcal{F}, \mathcal{G})$ sits as a closed subscheme of $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \times_T \mathcal{H}om(\mathcal{G}, \mathcal{F})$. $\square$

*Remark* 0.11. The above proof works in fair generality and we leave the reader to figure out the various situations in which these ideas can be used.

We are left with finding a presentation for $\mathrm{Coh}_{X/S}$. To this end we realise that the most obvious candidate for finding such a presentation is the Quot schemes of Grothendieck. For each $N > 0$ we denote by $\mathrm{Quot}_{\mathcal{O}_X^N/X/S}$ the scheme whose points are as below[1]

$$\mathrm{Quot}_{\mathcal{O}_X^N/X/S}(T) := \left\{ \text{Set of quotients of } \mathcal{O}_{X_T}^N \to \mathcal{M} \text{ such that } \mathcal{M} \text{ is flat over } T \right\}.$$

**Definition 0.12.** Consider the following conditions on a $T$ valued point of $\mathrm{Quot}_{\mathcal{O}_X^N/X/S}$ i.e. on a flat quotient $\alpha\colon \mathcal{O}_{X_T}^N \to \mathcal{M}$

(1) $R^p(f_T)_*\mathcal{M} = 0$ for all $p > 0$,
(2) $f_*\mathcal{O}_{X_T}^N = \mathcal{O}_T^N \to f_*\mathcal{M}$ is an isomorphism.

We denote the subset spanned by the quotients of the above form as $\mathrm{Quot}^\circ_{\mathcal{O}_X^N/X/S}$ and the next lemma shows that this is in fact an open (hence representable) subfunctor of the above.

**Lemma 0.13.** *The above conditions are functorial and open on the base. In particular* $\mathrm{Quot}^\circ_{\mathcal{O}_X^N/X/S}$ *is an open subfunctor of* $\mathrm{Quot}_{\mathcal{O}_X^N/X/S}$

*Proof.* We first check that the conditions are functorial on the base. There is a commutative diagram

$$\begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

and for any flat sheaf $F$ on $X$ we have the derived base change isomorphisms $Rf_*F_T \simeq Lg^*Rf_*F$ [Ill03, Thm. 3.2]. Thus $R^if_*F_T \simeq H^i(Lg^*\tau_{\geq i}Rf_*F)$ and so if $\tau_{\geq i}Rf_*F = 0$ for some $i$ then so is $R^jf_*F_T$ for all $j \geq i$. Thus the first condition is functorial. We want to see that the second condition is functorial as well. This follows because isomorphisms pullback to isomorphisms and (after the first condition has been imposed) the formation of $f_*F$ commutes with base change [Ill03, Rem. 3.11.2]. We have to check that for any $S$-scheme $T$ the left vertical arrow is an open immersion and is universally so.

$$\begin{tikzcd} T \times_{\mathrm{Quot}_{\mathcal{O}_X^N/X/S}} \mathrm{Quot}^\circ_{\mathcal{O}_X^N/X/S} \arrow[r] \arrow[d] & \mathrm{Quot}^\circ_{\mathcal{O}_X^N/X/S} \arrow[d] \\ T \arrow[r] & \mathrm{Quot}_{\mathcal{O}_X^N/X/S} \end{tikzcd}.$$

This amounts to checking that if $\alpha\colon \mathcal{O}_{X_T}^N \to \mathcal{L}$ is a quotient, then the *maximal*[2] locus $U \subseteq T$ with $R^i(f_U)_*\mathcal{L}$ is zero for all $i > 0$ and $\mathcal{O}_U^N \to f_*F$ is an isomorphism is open. For the first condition this follows from upper semicontinuity of rank of the cohomologies [Ill03, 3.13] for the second condition it follows from the observation that $f_*F$ is locally free and hence $\mathrm{Isom}(\mathcal{O}_S^N, f_*F)$ is an open subfunctor of $\mathrm{Hom}(\mathcal{O}_S^N, f_*F)$ since both inputs are locally free (so the maximal locus where the canonical map is an isomorphism is open) . $\square$

For the sequel we fix a relatively $f$-very ample line bundle $\mathcal{O}_X(1)$ on $X$.

Now for each $N, n > 0$ we define a morphism of stacks

$$P_{N,n}\colon \mathrm{Quot}^\circ_{\mathcal{O}_X^N/X/S} \to \mathrm{Coh}_{X/S}$$

defined by $P_{N,n}(\alpha\colon \mathcal{O}_X^N \to \mathcal{M}) = \mathcal{M}(-n)$.

[1] For the experts, we do not have any properness condition on the support since we are working with a projective morphism and hence the support of every coherent sheaf is proper.

[2] We thank Nathan Wenger for clarifying this point for us.

This gives us a morphism

$$P := (\coprod_{N,n} P_{N,n})\colon \coprod_{N,n} \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S} \to \mathrm{Coh}_{X/S}.$$

*Remark* 0.14. We explain why we need to twist by an $f$-ample line bundle. Suppose $S = \operatorname{Spec} k$ where $k$ is a field and $X = \mathbf{P}^n_k$. Then the bundle $\mathcal{O}(-1)$ is certainly flat but is not the quotient of $\mathcal{O}^N$ for any $N > 0$ since it has no global sections. However since upto twisting by $\mathcal{O}(1)$ every coherent sheaf has global sections we see that the morphism $P_{N,n}$ has some hope of being surjective.

**Lemma 0.15.** *The morphism $P$ is a surjective morphism of stacks.*

*Proof.* Unwinding definitions, we have to show for every $S$-scheme $T$ and for every $\mathcal{M} \in \mathrm{Coh}_{X/S}(T)$ there is an etale cover $T' \to T$ so that $\mathcal{M}|_{T'}$ is in the image of $P$.

To see this we first note that for any coherent sheaf $\mathcal{M}$ on $X_T$ there an $n > 0$ so that $R^i f_*\mathcal{M}(n) = 0$ for all $i > 0$. This follows from the projectivity of the morphism $f$ and the relative very ampleness of the twisting sheaf $\mathcal{O}_X(1)$. This shows that $f_*\mathcal{M}(n)$ is locally free on the base $T$. There is a Zariski cover $\coprod_i T_{n,N_i} \to T$ so that the pullback of $f_*\mathcal{M}(n)$ is free of rank $N_i$ on the $T_{n,N_i}$. We may as well assume $f_*\mathcal{M}(n)$ is free of rank $N$ on $T$. Then we have the evaluation map $f^*f_*\mathcal{M}(n) \to \mathcal{M}(n)$. Since $f_*\mathcal{M}(n)$ is free of rank $N$ we see that $f^*f_*\mathcal{M}(n) \simeq \mathcal{O}^N_{X_T}$ and we get a map $\mathcal{O}^N_{X_T} \to \mathcal{M}(n)$ which by relative global generation of the target is surjective. We need to check that the pushforward is an isomorphism. We have $\alpha\colon f_*f^*f_*\mathcal{M}(n) \to f_*\mathcal{M}(n)$ and we can compose on the left to get $f_*\mathcal{M}(n) \xrightarrow{\beta} f_*f^*f_*\mathcal{M}(n) \xrightarrow{\alpha} f_*\mathcal{M}(n)$ so that $\alpha \circ \beta = id$ which shows that $\alpha$ is the an epimorphism of sheaves. Since the sheaves are vector bundles of the same rank, this has to be an isomorphism, and so we are done. $\square$

We now check that the morphism is smooth.

**Lemma 0.16.** *The morphism $P$ is a smooth morphism of stacks.*

*Proof.* We have to check that for any $T \to S$ the base change $T \times_{\mathrm{Coh}_{X/S}} (\coprod_{N,n} \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S}) \to T$ is a smooth morphism, the representability being guaranteed by the representability of the diagonal. Since the coproduct of a smooth morphism is a smooth morphism we can fix an $N$ and a $n$ and consider the morphism $P_{N,n}\colon \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S} \to \mathrm{Coh}_{X/S}$ and the base change $p_1\colon T \times_{\mathrm{Coh}_{X/S}} \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S} \to T$ and reduce to proving the smoothness of this morphism. First note that a $T$ valued point of $\mathrm{Coh}_{X/S}$ is an $\mathcal{O}_T$ flat coherent sheaf $\mathcal{F}$ on $X_T$. Then the functor of points of $T \times_{\mathrm{Coh}_{X/S}} \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S}$ classifies for any $T' \to T$ the set of quotients $\alpha\colon \mathcal{O}^N_{X'_T} \to \mathcal{F}_{T'}(n)$ and such that $(f_{T'})_*(\alpha)\colon \mathcal{O}^N_{T'} \to (f_{T'})_*\mathcal{F}(n)$ is an isomorphism. But this means that $T'$ should factorise through the locus $T_{n,N}$ as above and in particular (by looking at the identity) we have a factorisation $p_1\colon T \times_{\mathrm{Coh}_{X/S}} \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S} \to T_{n,N} \to T$. Over $T_{n,N}$ we see that the functor of points are precisely isomorphisms $\mathrm{Isom}(\mathcal{O}^N_{T_{n,N}}, (f_{T_{n,N}})_*\mathcal{F}_{T_{n,N}})$. The reason we are only left with the isomorphism condition is because surjectivity of the map $\alpha\colon \mathcal{O}^N_{X_{T_{n,N}}} \to \mathcal{F}_{T_{n,N}}$ is immediate from the end of our argument in the previous lemma.

As a consequence in the factorisation $p_1\colon T \times_{\mathrm{Coh}_{X/S}} \mathrm{Quot}^{\circ}_{\mathcal{O}_X^N/X/S} \to T_{n,N} \to T$ we see that the first map is a $GL_{N,T_{n,N}}$-torsor and the second is an open immersion and so the composite is smooth. Thus we win. $\square$